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The reflection coefficient for minimal model conformal defects from perturbation theory

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Abstract: We consider a class of conformal defects in Virasoro minimal models that have been defined as fixed points of the renormalisation group and calculate the leading contribution to the reflection coefficient for these defects. This requires several structure constants of the operator algebra of the defect fields, for which we present a derivation in detail. We compare our results with our recent work on conformal defects in the tricritical Ising model.

Keywords: Conformal Field Theory, Boundary Quantum Field Theory, Renormalization Group

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1 Introduction

Defects in two-dimensional systems have been studied for a long time, see eg [1, 2] and references therein. In conformal field theory, attention has been focused primarily on defects which preserve some or all of the conformal symmetry. If the defect lies along the real axis, this can be expressed in terms of the continuity of various quantities. If the holomorphic and anti-holomorphic components T and $\bar{T}$ of the stress-energy tensor are each separately continuous across the defect, it is said to be topological; if $T-\bar{T}$ vanishes on the defect, it is called reflecting or factorised and corresponds to some combination of conformal boundary conditions on the upper and lower half planes. These are both examples of the more general case of a conformal defect for which $T-\bar{T}$ is continuous across the defect.

The Virasoro minimal models are amongst the simplest and most well-studied conformal field theories. The boundary conditions and topological defects have been completely classified in [3] and further studied in [4]. The situation of more general conformal defects is much less clear. The conformal defects in the Ising model were classified in [5] (and in the much simpler Lee-Yang model in [2]), but in general the only results found are either perturbative or numerical [6]. More recently, we have also found exact expressions for conformal defects in the tricritical Ising model [7] (based on ideas in [8]).

There has also been a great deal of study of defects between different conformal field theories, with exact classifications in a few cases [2], exact proposals [9] for defects related to renormalisation group flows, and perturbative calculations [10].

One characteristic of a conformal defect is its transmission coefficient $\mathcal{T}$, or equivalently its reflection coefficient $\mathcal{R} = 1 - \mathcal{T}$, which was defined in [2]. These take the values $\mathcal{R} = 0$ for a topological defect and $\mathcal{R} = 1$ for a factorised defect, and $0 < \mathcal{R} < 1$ for a general conformal defect in a unitary theory [11].

The aim of this paper is to calculate the reflection coefficient for a class of conformal defects in Virasoro minimal models defined as the fixed points of the perturbative renormalisation group flows considered in [6], and to compare this with the values found in [7] for the tri-critical Ising model.

The structure of the paper is as follows. In section 2 we define the perturbed defects that we will consider, state their fixed points and outline the calculation of the reflection coefficient for these fixed points. For this calculation we need several of the structure constants of the operator algebra of the defect fields. These are given in [12, 13] in terms of topological field theory data but in section 3 we provide an alternative derivation of these constants in terms of bulk and boundary CFT data, extending the results of [14].

In section 4 we calculate the perturbative integrals we need. In section 5 we give the value of $\mathcal{R}$ at the fixed points in terms of the bulk and boundary CFT data; these do not rely on the details of the model and so could have more general applicability. We then specialise to the particular case of the $(r, 2)$ defects in minimal models. In section 6 we compare the perturbative results for $\mathcal{R}$ and the boundary entropy g with known results in the tri-critical Ising model. Finally we state our conclusions in section 7.

2 The $D_{(r,2)}$ defect and its perturbations

We will concern ourselves only with diagonal $M_{p,q}$ Virasoro minimal models, also known as the (A_{p-1}, A_{q-1}) invariant [15]. These are labelled by two co-prime integers (p, q); we shall take $p \geq 2$, $q \geq 5$. The model has $(p-1)(q-1)/2$ primary fields corresponding to the Virasoro highest weight representations which are labelled by two integers (r, s) with $(r, s) \simeq (p - r, q - s)$. We are going to be especially interested in the representation $(1, 3)$, and we will write $h = h_{13} = 2p/q - 1$.

The elementary topological defects for this model were classified in [3], and are labelled by the same representations of the Virasoro algebra as the bulk fields. The space of local fields on the defects is also known. If we label the representations by a, then a primary field on the defect is labelled by two representations (a, b) which give its properties under the holomorphic and anti-holomorphic copies of the Virasoro algebra (but see the comment below on the transformation rules for defect fields). The multiplicity M_{ab} of the primary field with labels (a, b) on the defect with label d (which is $\tilde{V}_{ab;d}{}^{d}$ in the notation of [3]) are given in terms of the Verlinde fusion numbers N_{abc} by

$$M_{ab} = \sum_e N_{dae} N_{deb} = \sum_f N_{ddf} N_{fab} \,. \tag{2.1}$$

From the formula (2.1), a general (r, s) defect has (for $s > 2$ and q large enough) one chiral field of weights $(h, 0)$, one field of weights $(0, h)$ and three fields of weight (h, h). A defect of type $(r, 2)$ is special in that it has one chiral field ϕ of conformal weights $(h, 0)$, one chiral field $\bar{\phi}$ of weights $(0, h)$, but only a two dimensional space of fields $\{\varphi_\alpha\}$ of weights (h, h).

Furthermore, the $(r, 2)$ topological defect can be constructed as the fusion $(r, 1)$ and $(1, 2)$ topological defects and the operator product algebra of fields of type $(a, b) = ((1, s)(1, s'))$ is unaffected by this fusion, in exactly the same way that the action of topological defects on boundaries leaves operator algebras invariant [16]. This means that when considering the algebra of fields generated by the set $\{\mathbf{1}, \phi, \bar{\phi}, \varphi_\alpha\}$, we can restrict attention to just the $(1, 2)$ defect.

The fact that there is a two-dimensional space of fields $\{\varphi_a\}$ on the $(r, 2)$ defects allows one to choose a canonical basis of these fields with special properties so that the analysis of the sewing constraints is correspondingly simpler. These sewing constraints have been solved in [14] for the $(1, 2)$ defect in the non-unitary Lee-Yang model, the (A_1, A_4) theory, in which $D_{(1,2)}$ is the only non-trivial defect and $\{\mathbf{1}, \phi, \bar{\phi}, \varphi_\alpha\}$ are the only non-trivial primary defect fields. In this paper we extend this analysis to the fields $\{\mathbf{1}, \phi, \bar{\phi}, \varphi_\alpha\}$ on defects of type $D_{(r,2)}$ in all the (A_p, A_q) models.

We are interested in the perturbations of the defect $D_{(r,2)}$ by a combination of the fields ϕ and $\bar{\phi}$,

$$S = \int \left(\lambda\phi(x) + \bar{\lambda}\bar{\phi}(x)\right) \mathrm{d}x\,, \tag{2.2}$$

where the parameters λ and $\bar{\lambda}$ are independent. This is a relevant perturbation if $h < 1$ which is the case if $p < q$.

One important question is that of the transformation properties of fields on a defect under a conformal transformation. We will use the conventions of [13] which imply that defect fields always transform with the absolute value of the derivative of the conformal map, even if they are "chiral" defect fields. This is possible because the defect defines a direction through the insertion point of the field (the tangent vector along the defect), and so a defect field can pick up an extra phase under a conformal transformation: this is chosen so that all defect fields transform with the absolute value of the derivative of the conformal map. This has the advantage of making the perturbation well-defined on defects that are closed loops and making the correlation function independent of the orientation of the defect at the location of the defect field (as one would expect if the defect is genuinely topological). The question remains whether this choice for the transformation law of "chiral" defect fields is unique: the corresponding situation for a boundary and boundary fields was considered by Runkel [17], and there seems no way to fix it a priori; we stick to the conventions of [13] here for the good reasons cited above.

The expectation values in the perturbed defect $D_{(r,2)}(\lambda, \bar{\lambda})$ are formally given by

$$\langle\, \mathcal{O} \,\rangle_{D_{(r,2)}(\lambda,\bar{\lambda})} = \langle\, \mathcal{O}\, \exp(-S) \,\rangle_{D_{(r,2)}}\,. \tag{2.3}$$

This is only formal since there may be UV divergences in the integrals when the insertion points of two fields ϕ or two fields $\bar{\phi}$ meet and IR divergences from integration along the whole real axis. This means that the general procedure of regularisation and renormalisation may be needed to given meaning to the expression (2.3). This is explained in

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Affleck and Ludwig [18] and applied by Recknagel et al. in [19] to the case of boundary perturbations of the unitary minimal models where $q = p + 1$.

As explained in [6], when $y = 1 - h$ is small and positive, the results of [19] can immediately be applied to the case of defects with the perturbation (2.2) with the prediction (from third order perturbation theory) of three conformal defects at the fixed points

$$\text{(i)} \quad \lambda = \lambda^*, \bar{\lambda} = 0 \tag{2.4}$$

$$\text{(ii)} \quad \lambda = 0, \bar{\lambda} = \lambda^* \tag{2.5}$$

$$\text{(iii)} \quad \lambda = \bar{\lambda} = \lambda^* \tag{2.6}$$

The fixed points (i) and (ii) can be identified as the defect $D_{(2,1)}$ (if $r = 2$) and (more generally) the superposition $D_{(r-1,1)} \oplus D_{(r+1,1)}$; the fixed point (iii) is a potential new conformal defect, denoted by C in [6] in the case of the perturbation of the defect $D_{(1,2)}$. The value of λ^* is given (to first order in y) by

$$\lambda^* = \frac{y}{C^{\phi}_{\phi\phi}}, \tag{2.7}$$

where $C^{\phi}_{\phi\phi}$ is the coefficient of the field ϕ in the operator product expansion of ϕ with itself (3.9). Since $C^{\phi}_{\phi\phi}$ depends on the normalisation of ϕ, so does the value λ^* but this will cancel in any physical quantities and in particular in our calculation of $\mathcal{R}$.

2.1 The perturbative calculation of the reflection and transmission coefficients

The transmission and reflection coefficients of a conformal defect along the real axis were defined in [2] as

$$\mathcal{R} = \frac{\langle T^1\overline{T}^1 + T^2\overline{T}^2\rangle}{\langle (T^1 + \overline{T}^2)(\overline{T}^1 + T^2)\rangle}, \quad \mathcal{T} = 1 - \mathcal{R} \tag{2.8}$$

where T^1 and $\overline{T}^1$ are inserted at the point iY on the upper half-plane, while T^2 and $\overline{T}^2$ are inserted at the point $-iY$. For the unperturbed topological defect,

$$\langle\, T^1\overline{T}^1 \,\rangle = \langle\, T^2\overline{T}^2 \,\rangle = 0\,, \quad \langle\, T^1T^2 \,\rangle = \langle\, \overline{T}^1\overline{T}^2 \,\rangle = \frac{c}{32Y^4}\,, \tag{2.9}$$

and so $\mathcal{R} = 0$ and $\mathcal{T} = 1$.

For the defect with perturbation (2.2), the expansion of the perturbed quantities using (2.3) gives

$$\begin{aligned}
\langle\, T^1\overline{T}^1 \,\rangle = {} & \frac{1}{4}\lambda^2\bar{\lambda}^2 \int \mathrm{d}x\,\mathrm{d}x'\,\mathrm{d}y\,\mathrm{d}y' \langle\, T(iY)\,\overline{T}(iY)\,\phi(x)\phi(x')\bar{\phi}(y)\bar{\phi}(y')\,\rangle \\
& - \frac{1}{24}\lambda^3\bar{\lambda}^2 \int \mathrm{d}x\,\mathrm{d}x'\,\mathrm{d}x''\,\mathrm{d}y\,\mathrm{d}y' \langle\, T(iY)\,\overline{T}(iY)\,\phi(x)\phi(x')\phi(x'')\bar{\phi}(y)\bar{\phi}(y')\,\rangle \\
& - \frac{1}{24}\lambda^2\bar{\lambda}^3 \int \mathrm{d}x\,\mathrm{d}x'\,\mathrm{d}y\,\mathrm{d}y'\,\mathrm{d}y'' \langle\, T(iY)\,\overline{T}(iY)\,\phi(x)\phi(x')\bar{\phi}(y)\bar{\phi}(y')\bar{\phi}(y'')\,\rangle \\
& + O(\lambda^6)\,,
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
\langle\, T^1T^2 \,\rangle = {} & \frac{c}{32Y^4} \\
& + \frac{1}{2}\lambda^2 \int \mathrm{d}x\,\mathrm{d}x' \langle\, T(iY)\,T(-iY)\,\phi(x)\phi(x')\,\rangle \\
& + \frac{1}{2}\bar{\lambda}^2 \int \mathrm{d}y\,\mathrm{d}y' \langle\, T(iY)\,T(-iY)\,\bar{\phi}(y)\bar{\phi}(y')\,\rangle + O(\lambda^3)\,,
\end{aligned} \tag{2.11}$$

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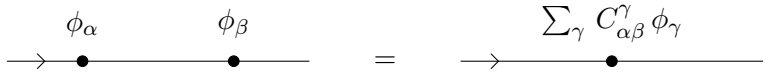

Figure 1. The OPE of defect fields.

and so to find the leading order term in $\mathcal{R}$, we only need to calculate the first term in $\langle\, T^1\overline{T}^1\,\rangle$ and $\langle\, T^2\overline{T}^2\,\rangle$. It turns out there are neither UV nor IR divergences in these integrals, their dependence on Y is simply Y^{-4} and the reflection coefficient $\mathcal{R}$ (to leading order) is indeed independent of Y as expected. We shall take $Y = 1$ from now on.

The consequence is that the only correlation function we need to evaluate is

$$\langle\, T(i)\overline{T}(i)\phi(x)\phi(x')\bar{\phi}(y)\bar{\phi}(y')\,\rangle\,, \tag{2.12}$$

where the insertion points can be in any order. This is equal to

$$\langle\, T(-i)\overline{T}(-i)\phi(-x)\phi(-x')\bar{\phi}(-y)\bar{\phi}(-y')\,\rangle\,, \tag{2.13}$$

by rotation through π.

The analytic structure is simple,

$$\left\langle\, T(i)\,\overline{T}(i)\,\phi(x)\phi(x')\bar{\phi}(y)\bar{\phi}(y')\,\right\rangle = \mathcal{C}\,\frac{(x'-x)^{2-2h}\,(y'-y)^{2-2h}}{(i-x)^2(i-x')^2(i+y)^2(i+y')^2}\,, \tag{2.14}$$

but the constant $\mathcal{C}$ depends on the order of the insertion points $\{x, x', y, y'\}$ and is determined by the operator algebra structure constants, so we now turn to the calculation of some of the structure constants of the local fields on the defect $D_{(r,2)}$.

3 The structure constants

In this section we will calculate some structure constants for the $(r, 2)$ defect in the diagonal Virasoro Minimal models. These structure constants can be found in terms of topological field theory data [12, 13] which is a general method allowing one to find all the structure constants in the defect theory, but we will not use it here and instead only use elementary properties of the conformal field theory to find the particular structure constants we need for the perturbative calculation of the reflection coefficient in the minimal models. While the calculation is motivated by the $(r, 2)$ minimal model defects, the results will be expressed in terms of the bulk and boundary CFT data and so are applicable to any defect with the same fusion rules.

We note here that we will use the conventions of [13] so that the structure constant $C^\gamma_{\alpha\beta}$ is the coefficient of the field ϕ_γ appearing in the OPE of the fields $\phi_\alpha(x)$ with $\phi_\beta(y)$ on the defect oriented opposite to the real line with $x > y$, which means that this coefficient appears in the OPE of the fields ϕ_α with ϕ_β as they appear along the defect. Rotating by π, we obtain the picture in figure 1.

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3.1 The bulk theory

The (A_{p-1}, A_{q-1}) Virasoro minimal model has $(p-1)(q-1)/2$ bulk primary fields, of which we are especially interested in the field φ of type $(1,3)$. If we set $t = p/q$, then

$$h_{1,3} = h = 2t - 1\,, \tag{3.1}$$

and $h < 1$ if $t < 1$, that is $p < q$.

The fusion rules for this field are

$$[\varphi] \star [\varphi] = [1] + [\varphi] + [\chi]\,, \tag{3.2}$$

where χ is of type (1,5) and has conformal weights (h', h') where $h' = h_{1,5} = 6t-2$. Hence, the OPE of φ with itself is

$$\varphi(z,\bar{z})\varphi(w,\bar{w}) = \frac{d_{\varphi\varphi}}{|z-w|^{4h}} + \frac{C^{\varphi}_{\varphi\varphi}\,\varphi(w,\bar{w})}{|z-w|^{2h}} + \frac{C^{\chi}_{\varphi\varphi}\,\chi(w,\bar{w})}{|z-w|^{4h-2h'}} + \dots \tag{3.3}$$

The structure constant $C^{\varphi}_{\varphi\varphi}$ clearly depends on the choice of $d_{\varphi\varphi}$ (see [20, 21] for different conventions) but the combination

$$\frac{(C^{\varphi}_{\varphi\varphi})^2}{d_{\varphi\varphi}} \tag{3.4}$$

is independent of the normalisation. It takes the value

$$\frac{(C^{\varphi}_{\varphi\varphi})^2}{d_{\varphi\varphi}} = -(1-2t)^2 \frac{\Gamma(2-3t)}{\Gamma(3t-1)} \frac{\Gamma(4t-1)^2}{\Gamma(2-4t)^2} \frac{\Gamma(t)^3}{\Gamma(1-t)^3} \frac{\Gamma(1-2t)^4}{\Gamma(2t)^4}\,. \tag{3.5}$$

We are interested in the small y limit, with $h = 1 - y$ in which case

$$\frac{(C^{\varphi}_{\varphi\varphi})^2}{d_{\varphi\varphi}} = \frac{16}{3} - 16y + O(y)^2\,. \tag{3.6}$$

3.2 The defect theory

The defects of the (A_{p-1}, A_{q-1}) Virasoro models are not intrinsically oriented, but the operator product of fields along the defect depends on the ordering of the fields; we shall assume that we can define an orientation for the defects but that all results will be independent of this orientation.

Since the space of fields $\{\varphi_\alpha\}$ of weights (h,h) is only two-dimensional for a defect of type $(r,2)$, we can take as a basis the fields φ_L and φ_R which are the limits of the bulk field φ as it approaches the defect from the left or the right respectively as one looks along the defects — see figure 2.

Note that the operator product algebra of the fields $\{\mathbf{1}, \phi, \bar{\phi}, \varphi_L, \varphi_R\}$ does not close on these fields, other fields can arise as well, namely fields with weights (h,h'), (h',h) and (h',h') which we denote by ψ, $\bar{\psi}$ and $\{\chi_L, \chi_R\}$ (which again are the limits of the field $\chi(z,\bar{z})$ as it approaches the defect from the left and the right). Although we should mention the existence of these fields and their occurrence in the operator products of some of the fields $\{\phi, \bar{\phi}, \varphi_\alpha\}$, we will not need any of the structure constants including these fields as they will not contribute to any of the sewing constraints considered later on.

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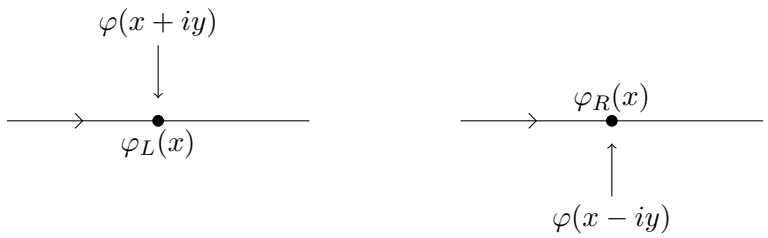

Figure 2. The fields φ_L and φ_R defined as limits of the bulk field.

Φ_a	h_a	$\bar{h}_a$
$\mathbf{1}$	0	0
ϕ	h	0
$\bar{\phi}$	0	h
φ_α	h	h
ψ	h'	h
$\bar{\psi}$	h	h'
χ_α	h'	h'

(3.7)

Table 1. Some of the primary fields occurring on the defect $(r, 2)$.

We use the generic labels $\{a, b, \ldots\}$ for all of these fields and the labels $\{\alpha, \beta, \ldots\}$ for the set $\{L, R\}$. The conformal weights of the field Φ_a are $(h_a, \bar{h}_a)$ as in table 1.

We now define the structure constants between these fields from their operator product expansions (we show the possibility of fields $\{\psi, \bar{\psi}, \chi_\alpha\}$ appearing in an OPE by placing the fields in square brackets [ ]).

If both fields are chiral, there are 8 structure constants $\{d_{\phi\phi}, d_{\bar{\phi}\bar{\phi}}, C^{\phi}_{\phi\phi}, C^{\bar{\phi}}_{\bar{\phi}\bar{\phi}}, C^{\alpha}_{\phi\bar{\phi}}, C^{\alpha}_{\bar{\phi}\phi}\}$ appearing in the OPEs (recall here that x and y are ordered along the defect):

$$\phi(x)\phi(y) = \frac{d_{\phi\phi}}{|x-y|^{2h}} + \frac{C^{\phi}_{\phi\phi}\,\phi(y)}{|x-y|^{h}} + \ldots \tag{3.8}$$

$$\bar{\phi}(x)\bar{\phi}(y) = \frac{d_{\bar{\phi}\bar{\phi}}}{|x-y|^{2h}} + \frac{C^{\bar{\phi}}_{\bar{\phi}\bar{\phi}}\,\bar{\phi}(y)}{|x-y|^{h}} + \ldots\,, \tag{3.9}$$

$$\phi(x)\bar{\phi}(y) = C^{L}_{\phi\bar{\phi}}\,\varphi_L(x,y) + C^{R}_{\phi\bar{\phi}}\,\varphi_R(x,y) + \ldots\,, \tag{3.10}$$

$$\bar{\phi}(x)\phi(y) = C^{L}_{\bar{\phi}\phi}\,\varphi_L(y,x) + C^{R}_{\bar{\phi}\phi}\,\varphi_R(y,x) + \ldots\,. \tag{3.11}$$

With one chiral field on the left, there are 12 structure constants $\{C^{\bar{\phi}}_{\phi\alpha}, C^{\phi}_{\bar{\phi}\alpha}, C^{\beta}_{\phi\alpha}, C^{\beta}_{\bar{\phi}\alpha}\}$ in the OPEs

$$\phi(x)\varphi_\alpha(z,\bar{z}) = \frac{C^{\bar{\phi}}_{\phi\alpha}\,\bar{\phi}(\bar{z})}{|x-z|^{2h}} + \frac{C^{L}_{\phi\alpha}\,\varphi_L(z,\bar{z})}{|x-z|^{h}} + \frac{C^{R}_{\phi\alpha}\,\varphi_R(z,\bar{z})}{|x-z|^{h}} + [\psi] + \ldots\,, \tag{3.12}$$

$$\bar{\phi}(x)\varphi_\alpha(z,\bar{z}) = \frac{C^{\phi}_{\bar{\phi}\alpha}\,\phi(z)}{|x-\bar{z}|^{2h}} + \frac{C^{L}_{\bar{\phi}\alpha}\,\varphi_L(z,\bar{z})}{|x-\bar{z}|^{h}} + \frac{C^{R}_{\bar{\phi}\alpha}\,\varphi_R(z,\bar{z})}{|x-\bar{z}|^{h}} + [\bar{\psi}] + \ldots\,. \tag{3.13}$$

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likewise there are 12 structure constants $\{C^{\bar\phi}_{\alpha\phi}, C^{\phi}_{\alpha\bar\phi}, C^{\beta}_{\alpha\phi}, C^{\beta}_{\alpha\bar\phi}\}$ in the OPEs with one field chiral on the right:

$$\varphi_\alpha(z,\bar z)\phi(x) = \frac{C^{\bar\phi}_{\alpha\phi}\,\bar\phi(\bar z)}{|z-x|^{2h}} + \frac{C^{L}_{\alpha\phi}\,\varphi_L(z,\bar z)}{|z-x|^{h}} + \frac{C^{R}_{\alpha\phi}\,\varphi_R(z,\bar z)}{|z-x|^{h}} + [\psi] + \dots\,, \tag{3.14}$$

$$\varphi_\alpha(z,\bar z)\bar\phi(x) = \frac{C^{\phi}_{\alpha\bar\phi}\,\phi(z)}{|\bar z-x|^{2h}} + \frac{C^{L}_{\alpha\bar\phi}\,\varphi_L(z,\bar z)}{|\bar z-x|^{h}} + \frac{C^{R}_{\alpha\bar\phi}\,\varphi_R(z,\bar z)}{|\bar z-x|^{h}} + [\bar\psi] + \dots\,. \tag{3.15}$$

Finally there are 20 structure constants $\{d_{\alpha\beta}, C^{\phi}_{\alpha\beta}, C^{\bar\phi}_{\alpha\beta}, C^{\gamma}_{\alpha\beta}\}$ in the OPEs involving no chiral fields:

$$\begin{aligned}\varphi_\alpha(z,\bar z)\varphi_\beta(w,\bar w) = {}& \frac{d_{\alpha\beta}}{|z-w|^{4h}} + \frac{C^{\phi}_{\alpha\beta}\phi(w)}{|z-w|^{h}|\bar z-\bar w|^{2h}} + \frac{C^{\bar\phi}_{\alpha\beta}\bar\phi(\bar w)}{|\bar z-\bar w|^{h}|z-w|^{2h}} \\ & + \frac{C^{L}_{\alpha\beta}\varphi_L(w,\bar w)}{|z-w|^{2h}} + \frac{C^{R}_{\alpha\beta}\varphi_R(w,\bar w)}{|z-w|^{2h}} + [\psi,\bar\psi,\chi_\alpha] + \dots\,.\end{aligned} \tag{3.16}$$

Having defined the fifty-two structure constants we need to calculate, we now set about finding relations. The simplest come from the fact that the orientation of the defect is in fact not physical.

3.3 Symmetry relations

Since the defect is not intrinsically oriented, our labelling over-counts the structure constants: sixteen constants are related by changing the orientation of the defect, as follows:

$$C^{L}_{\phi\bar\phi} = C^{R}_{\bar\phi\phi}\,,\quad C^{R}_{\phi\bar\phi} = C^{L}_{\bar\phi\phi}\,,\quad d_{LL} = d_{RR}\,,\quad d_{LR} = d_{RL}\,, \tag{3.17}$$

$$C^{L}_{LL} = C^{R}_{RR}\,,\ C^{R}_{LL} = C^{L}_{RR}\,,\ C^{L}_{LR} = C^{R}_{RL}\,,\ C^{L}_{RL} = C^{R}_{LR}\,. \tag{3.18}$$

$$C^{R}_{\phi R} = C^{L}_{L\phi}\,,\quad C^{L}_{\phi R} = C^{R}_{L\phi}\,,\quad C^{R}_{\phi L} = C^{L}_{R\phi}\,,\quad C^{L}_{\phi L} = C^{R}_{R\phi}\,, \tag{3.19}$$

$$C^{R}_{\bar\phi R} = C^{L}_{L\bar\phi}\,,\quad C^{L}_{\bar\phi R} = C^{R}_{L\bar\phi}\,,\quad C^{R}_{\bar\phi L} = C^{L}_{R\bar\phi}\,,\quad C^{L}_{\bar\phi L} = C^{R}_{R\bar\phi}\,. \tag{3.20}$$

3.3.1 Bulk field relations

We can use the fact that φ_L and φ_R are the limits of bulk fields to find d_{LL}, d_{LR}, d_{RL} and d_{RR}, as well as C^L_{LL}, C^R_{LL}, C^L_{RR} and C^R_{RR}.

In the bulk, we have (3.3). Bringing this OPE towards a defect from the left, we obtain

$$d_{LL} = d_{\varphi\varphi}\,,\quad C^{L}_{LL} = C^{\varphi}_{\varphi\varphi}\,,\quad C^{R}_{LL} = C^{\phi}_{LL} = C^{\bar\phi}_{LL} = 0\,. \tag{3.21}$$

We have also found that

$$C^{\chi_L}_{LL} = C^{\chi}_{\varphi\varphi}\,,\quad C^{\chi_R}_{LL} = C^{\psi}_{LL} = C^{\bar\psi}_{LL} = 0\,, \tag{3.22}$$

but these four constants are not of interest to us.

Likewise, bringing the bulk OPE (3.3) towards a defect from the right, we obtain

$$d_{RR} = d_{\varphi\varphi}\,,\quad C^{R}_{RR} = C^{\varphi}_{\varphi\varphi}\,,\quad C^{L}_{RR} = C^{\phi}_{RR} = C^{\bar\phi}_{RR} = 0\,. \tag{3.23}$$

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We can relate d_{LR} to d_{LL} using the expansion of a defect operator as a linear combination of projectors onto the various sectors of the bulk theory,[1]

$$\hat{D}_a = \sum_b \gamma_{ab} \hat{P}_b \,, \tag{3.24}$$

to get

$$d_{LR} = \frac{\langle\varphi|\ \hat{D}_a\ |\varphi\rangle}{\langle 0|\ \hat{D}_a\ |0\rangle} = \frac{\gamma_{a\varphi}}{\gamma_{a0}} \frac{\langle\varphi|\varphi\rangle}{\langle 0|0\rangle} = \frac{\gamma_{a\varphi}}{\gamma_{a0}}\, d_{LL} \equiv \gamma\, d_{LL} \,, \tag{3.25}$$

where γ is given in terms of the coefficients γ_{ab} in the defect operator expansion. Using the expression for the minimal model defects in terms of projectors given in [3],

$$\hat{D}_{r,2} = \sum_{r',s} \frac{S_{(r,2),(r',s)}}{S_{(1,1),(r',s)}} \hat{P}_{r',s} \,, \tag{3.26}$$

where $S_{(rs)(r's')}$ is the modular S-matrix given in the appendix, we find

$$d_{LR} = \gamma\, d_{LL} \,, \quad \gamma = 2\cos(2\pi t) - 1 \,, \tag{3.27}$$

which is independent of r, as expected.

3.4 Defect — boundary identification

We next use the fact that the OPE algebra of ϕ along the real axis is the same as that of the boundary field on the $(r,2)$ boundary — we obtain this identification by bringing the $(r,2)$ defect next to the identity boundary as considered in [16]. Likewise, the algebra of $\bar{\phi}$ is also the same as the boundary algebra.

This means that

$$d_{\phi\phi} = d_{\bar{\phi}\bar{\phi}} \,, \quad C^{\phi}_{\phi\phi} = C^{\bar{\phi}}_{\bar{\phi}\bar{\phi}} \,, \tag{3.28}$$

and these are given by the structure constants of the boundary CFT. These constants depend on the normalisation of the fields and there is one convention-independent combination,

$$\frac{(C^{\phi}_{\phi\phi})^2}{d_{\phi\phi}} \,. \tag{3.29}$$

Using Runkel's solution to the boundary algebra [21], the result is

$$\frac{(C^{\phi}_{\phi\phi})^2}{d_{\phi\phi}} = \frac{\Gamma(2-3t)\Gamma(t)\Gamma(1-2t)^3}{\Gamma(2-4t)^2\Gamma(-1+2t)\Gamma(1-t)^2} \,, \tag{3.30}$$

which has the small y expansion

$$\frac{(C^{\phi}_{\phi\phi})^2}{d_{\phi\phi}} = \frac{8}{3} - 4y + O(y^2) \,. \tag{3.31}$$

Note that the structure constant again does not depend on r.

[1] In general a defect is a sum of maps between equivalent pairs of left and right representations, but if each pair of representations appears at most once then these are simply projectors.

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3.4.1 Three-point function constraints

We can express the three point function

$$\langle \Phi_a(u)\Phi_b(v)\Phi_c(w)\rangle\,, \tag{3.32}$$

in two different ways, using the OPE of Φ_a with Φ_b first, or instead using the OPE of Φ_b with Φ_c first, leading to the constraint

$$\sum_e C_{ab}^e d_{ec} = \sum_f d_{af} C_{bc}^f\,. \tag{3.33}$$

Taking a and c chiral, this gives the simple relations

$$C_{\phi R}^{\bar\phi} d_{\bar\phi\bar\phi} = C_{R\bar\phi}^{\phi} d_{\phi\phi}\,,\quad C_{\phi L}^{\bar\phi} d_{\bar\phi\bar\phi} = C_{L\bar\phi}^{\phi} d_{\phi\phi}\,, \tag{3.34}$$

$$C_{\bar\phi R}^{\phi} d_{\phi\phi} = C_{R\phi}^{\bar\phi} d_{\bar\phi\bar\phi}\,,\quad C_{\bar\phi L}^{\phi} d_{\phi\phi} = C_{L\phi}^{\bar\phi} d_{\bar\phi\bar\phi}\,, \tag{3.35}$$

which, using (3.28) become

$$C_{\phi R}^{\bar\phi} = C_{R\bar\phi}^{\phi}\,,\quad C_{\phi L}^{\bar\phi} = C_{L\bar\phi}^{\phi}\,,\quad C_{\bar\phi R}^{\phi} = C_{R\phi}^{\bar\phi}\,,\quad C_{\bar\phi L}^{\phi} = C_{L\phi}^{\bar\phi}\,. \tag{3.36}$$

Taking only a chiral and the two non-chiral fields equal, this gives the slightly more complicated

$$C_{\phi R}^{R} d_{RR} + C_{\phi R}^{L} d_{LR} = C_{RR}^{\phi} d_{\phi\phi} = 0\,,\quad C_{\bar\phi R}^{R} d_{RR} + C_{\bar\phi R}^{L} d_{LR} = C_{RR}^{\bar\phi} d_{\bar\phi\bar\phi} = 0\,, \tag{3.37}$$

$$C_{\phi L}^{R} d_{RL} + C_{\phi L}^{L} d_{LL} = C_{LL}^{\phi} d_{\phi\phi} = 0\,,\quad C_{\bar\phi L}^{R} d_{RL} + C_{\bar\phi L}^{L} d_{LL} = C_{LL}^{\bar\phi} d_{\bar\phi\bar\phi} = 0\,, \tag{3.38}$$

which using (3.25) become

$$C_{\phi R}^{R} = -\gamma C_{\phi R}^{L}\,, C_{\bar\phi R}^{R} = -\gamma C_{\bar\phi R}^{L}\,,\quad C_{\phi L}^{L} = -\gamma C_{\phi L}^{R}\,, C_{\bar\phi L}^{L} = -\gamma C_{\bar\phi L}^{R}\,. \tag{3.39}$$

Taking a chiral and the other two fields different, we get

$$C_{LR}^{\phi} d_{\phi\phi} = d_{LL} C_{R\phi}^{L} + d_{LR} C_{R\phi}^{R}\,,\quad C_{LR}^{\bar\phi} d_{\bar\phi\bar\phi} = d_{LL} C_{R\bar\phi}^{L} + d_{LR} C_{R\bar\phi}^{R}\,, \tag{3.40}$$

$$C_{RL}^{\phi} d_{\phi\phi} = d_{RR} C_{R\phi}^{R} + d_{RL} C_{R\phi}^{L}\,,\quad C_{RL}^{\bar\phi} d_{\bar\phi\bar\phi} = d_{RR} C_{R\bar\phi}^{R} + d_{RL} C_{R\bar\phi}^{L}\,. \tag{3.41}$$

Using $d_{LR} = \gamma d_{\varphi\varphi}$, these become

$$C_{LR}^{\phi} = \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(C_{R\phi}^{L} + \gamma C_{R\phi}^{R})\,,\quad C_{LR}^{\bar\phi} = \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(C_{R\bar\phi}^{L} + \gamma C_{R\bar\phi}^{R})\,, \tag{3.42}$$

$$C_{RL}^{\phi} = \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(C_{L\phi}^{R} + \gamma C_{L\phi}^{L})\,,\quad C_{RL}^{\bar\phi} = \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(C_{L\bar\phi}^{R} + \gamma C_{L\bar\phi}^{L})\,. \tag{3.43}$$

Finally, taking only b chiral, we get

$$C_{R\phi}^{\bar\phi} d_{\bar\phi\bar\phi} = d_{RR} C_{\phi\bar\phi}^{R} + d_{RL} C_{\phi\bar\phi}^{L}\,,\quad C_{L\phi}^{\bar\phi} d_{\bar\phi\bar\phi} = d_{LR} C_{\phi\bar\phi}^{R} + d_{LL} C_{\phi\bar\phi}^{L}\,, \tag{3.44}$$

$$C_{R\bar\phi}^{\phi} d_{\phi\phi} = d_{RR} C_{\bar\phi\phi}^{R} + d_{RL} C_{\bar\phi\phi}^{L}\,,\quad C_{L\bar\phi}^{\phi} d_{\phi\phi} = d_{LR} C_{\bar\phi\phi}^{R} + d_{LL} C_{\bar\phi\phi}^{L}\,. \tag{3.45}$$

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Looking at the first of these, it becomes

$$\begin{aligned} C^{\bar{\phi}}_{R\phi} &= \frac{1}{d_{\bar{\phi}\bar{\phi}}}(d_{RR}C^{R}_{\phi\bar{\phi}} + d_{RL}C^{L}_{\phi\bar{\phi}}) \\ &= \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(C^{R}_{\phi\bar{\phi}} + \gamma C^{L}_{\phi\bar{\phi}}) \\ &= \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(C^{R}_{\phi\bar{\phi}} + \gamma C^{R}_{\bar{\phi}\phi})\,. \end{aligned} \tag{3.46}$$

Likewise we get

$$C^{\bar{\phi}}_{L\phi} = \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(\gamma C^{R}_{\phi\bar{\phi}} + C^{R}_{\bar{\phi}\phi})\,, \tag{3.47}$$

$$C^{\phi}_{R\bar{\phi}} = \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(C^{R}_{\bar{\phi}\phi} + \gamma C^{R}_{\phi\bar{\phi}})\,, \tag{3.48}$$

$$C^{\phi}_{L\bar{\phi}} = \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(\gamma C^{R}_{\bar{\phi}\phi} + C^{R}_{\phi\bar{\phi}})\,, \tag{3.49}$$

which also imply

$$C^{\phi}_{R\bar{\phi}} = C^{\bar{\phi}}_{L\phi}\,,\quad C^{\phi}_{L\bar{\phi}} = C^{\bar{\phi}}_{R\phi}\,. \tag{3.50}$$

3.4.2 Bulk field expectation operator product

To find C^{R}_{LR} we use the inner product matrix $d_{\alpha\beta}$ of defect fields φ_L and φ_R and cyclicity of the three point constant $C_{\alpha\beta\gamma}$ defined by

$$\langle\varphi_\alpha(u,\bar{u})\varphi_\beta(v,\bar{v})\varphi_\gamma(w,\bar{w})\rangle = C_{\alpha\beta\gamma}\left(|u-v||v-w||v-w|\right)^{-2h}\,. \tag{3.51}$$

Using $C^{\gamma}_{\alpha\beta} = d^{\gamma\epsilon}C_{\alpha\beta\epsilon}$ and $C_{\alpha\beta\gamma} = C_{\gamma\beta\alpha}$ and the relations (3.21) and (3.23), we get

$$\begin{aligned} C^{R}_{LR} &= d^{RR}C_{LRR} + d^{RL}C_{LRL} \\ &= d^{RR}C_{RRL} + d^{RL}C_{LLR} \\ &= d^{RR}(d_{LL}C^{L}_{RR} + d_{LR}C^{R}_{RR}) + d^{RL}(d_{RL}C^{L}_{LL} + d_{RR}C^{R}_{LL}) \\ &= (d^{RR}d_{LR} + d^{RL}d_{RL})C^{\varphi}_{\varphi\varphi} \\ &= (d^{RR} + d^{RL})d_{RL}C^{\varphi}_{\varphi\varphi}\,. \end{aligned} \tag{3.52}$$

With the inner-product matrix $d_{\alpha\beta} = \langle\varphi_\alpha|\varphi_\beta\rangle$,

$$d_{\alpha\beta} = \begin{pmatrix} d_{LL} & d_{LR} \\ d_{RL} & d_{RR} \end{pmatrix} = d_{\varphi\varphi}\begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix}\,, \tag{3.53}$$

and its inverse

$$d^{\alpha\beta} = \begin{pmatrix} d^{LL} & d^{LR} \\ d^{RL} & d^{RR} \end{pmatrix} = \frac{1}{d_{\varphi\varphi}(1-\gamma^2)}\begin{pmatrix} 1 & -\gamma \\ -\gamma & 1 \end{pmatrix}\,, \tag{3.54}$$

we obtain

$$C^{R}_{LR} = \frac{\gamma}{1+\gamma}C^{\varphi}_{\varphi\varphi}\,. \tag{3.55}$$

Likewise, we find all four of these structure constants are equal,

$$C^{R}_{RL} = C^{L}_{LR} = C^{L}_{RL} = C^{R}_{LR} = \frac{\gamma}{1+\gamma}C^{\varphi}_{\varphi\varphi}\,. \tag{3.56}$$

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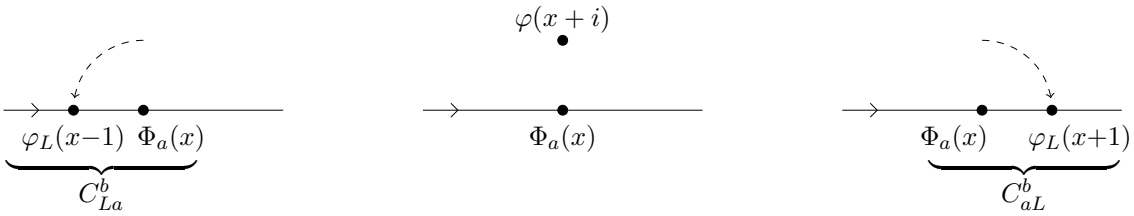

Figure 3. The relation between C^b_{La} and C^b_{aL} from continuity in the bulk.

3.4.3 Continuity of bulk fields

We can relate the structure constants C^b_{aL} and C^b_{La} by moving the insertion point of the field φ_L from the right of the field a to the left through the bulk. If the defect is oriented along the x axis in the plane, then the field φ_L can be moved through the upper half plane, as in figure 3.

Likewise, we can relate C^b_{aR} and $C^b_{R,a}$ by moving the field φ_R through the lower half plane.

Since the OPEs of the bulk field φ and the defect field φ_L with Φ_a are

$$\phi_a(u,\bar{u})\varphi(z,\bar{z}) = C^b_{a\varphi}\Phi_b(u,\bar{u})(u-z)^{h_b-h_a-h}(\bar{u}-\bar{z})^{\bar{h}_b-\bar{h}_a-h} + \dots\,, \tag{3.57}$$

$$\phi_a(u,\bar{u})\varphi_L(z,\bar{z}) = C^b_{aL}\Phi_b(u,\bar{u})|u-z|^{h_b-h_a-h}|\bar{u}-\bar{z}|^{\bar{h}_b-\bar{h}_a-h} + \dots\,, \tag{3.58}$$

$$\varphi_L(z,\bar{z})\phi_a(u,\bar{u}) = C^b_{La}\Phi_b(u,\bar{u})|z-u|^{h_b-h_a-h}|\bar{z}-\bar{u}|^{\bar{h}_b-\bar{h}_a-h} + \dots\,, \tag{3.59}$$

we get the relations

$$C^b_{La} = \exp(i\pi(h_b-\bar{h}_b-h_a+\bar{h}_a))C^b_{aL}\,, \tag{3.60}$$

$$C^b_{Ra} = \exp(-i\pi(h_b-\bar{h}_b-h_a+\bar{h}_a))C^b_{aR}\,. \tag{3.61}$$

We again list the cases according to the number of chiral fields involved:

- No chiral fields: we find identities consistent with equation (3.56)

$$C^R_{LR} = C^R_{RL}\,,\quad C^L_{LR} = C^L_{RL}\,. \tag{3.62}$$

- If Φ_b is chiral and Φ_a is not; with $\zeta = \exp(i\pi h)$:

$$C^{\phi}_{L\alpha} = \zeta C^{\phi}_{\alpha L}\,,\quad C^{\bar{\phi}}_{L\alpha} = \zeta^{-1}C^{\bar{\phi}}_{\alpha L}\,,\quad C^{\phi}_{R\alpha} = \zeta^{-1}C^{\phi}_{\alpha R}\,,\quad C^{\bar{\phi}}_{R\alpha} = \zeta C^{\bar{\phi}}_{\alpha R}\,, \tag{3.63}$$

  and hence

$$C^{\phi}_{LL} = C^{\bar{\phi}}_{LL} = C^{\phi}_{RR} = C^{\bar{\phi}}_{RR} = 0\,,\quad C^{\phi}_{LR} = \zeta C^{\phi}_{RL}\,,\quad C^{\bar{\phi}}_{LR} = \zeta^{-1}C^{\bar{\phi}}_{RL}\,, \tag{3.64}$$

  where the first four structure constants were already found to be zero in equations (3.21) and (3.23).

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- If Φ_a is chiral and Φ_b is not:

$$C^L_{L\phi} = \zeta^{-1} C^L_{\phi L}\,, \quad C^R_{L\phi} = \zeta^{-1} C^R_{\phi L}\,, \quad C^L_{L\bar\phi} = \zeta C^L_{\bar\phi L}\,, \qquad C^R_{L\bar\phi} = \zeta C^R_{\bar\phi L}\,, \tag{3.65}$$

$$C^L_{R\phi} = \zeta C^L_{\phi R}\,, \qquad C^R_{R\phi} = \zeta C^R_{\phi R}\,, \qquad C^L_{R\bar\phi} = \zeta^{-1} C^L_{\bar\phi R}\,, \quad C^R_{R\bar\phi} = \zeta^{-1} C^R_{\bar\phi R}\,, \tag{3.66}$$

$$C^\phi_{LR} = \zeta C^\phi_{RL}\,, \qquad C^{\bar\phi}_{LR} = \zeta^{-1} C^{\bar\phi}_{RL}\,. \tag{3.67}$$

- If both Φ_a and Φ_b are chiral:

$$C^{\bar\phi}_{L\phi} = \zeta^{-2} C^{\bar\phi}_{\phi L}\,, \quad C^{\phi}_{L\bar\phi} = \zeta^{2} C^{\phi}_{\bar\phi L}\,, \quad C^{\bar\phi}_{R\phi} = \zeta^{2} C^{\bar\phi}_{\phi R}\,, \quad C^{\phi}_{R\bar\phi} = \zeta^{-2} C^{\phi}_{\bar\phi R}\,. \tag{3.68}$$

3.5 Unknown constants

We summarise the results so far, distinguishing the structure constants by the number of chiral fields they involve.

3.5.1 No chiral fields

These are all known in terms of the bulk field data:

$$d_{RR} = d_{LL} = d_{\varphi\varphi}\,, \qquad d_{LR} = d_{RL} = \gamma\, d_{\varphi\varphi}\,, \tag{3.69}$$

$$C^L_{LL} = C^R_{RR} = C^\varphi_{\varphi\varphi}\,, \qquad C^R_{LL} = C^L_{RR} = 0\,, \tag{3.70}$$

$$C^R_{LR} = C^L_{LR} = C^R_{RL} = C^L_{RL} = \frac{\gamma}{1+\gamma}\, C^\varphi_{\varphi\varphi}\,. \tag{3.71}$$

3.5.2 Three chiral fields

These are also all known in terms of the boundary field theory data [21]:

$$C^{\bar\phi}_{\phi\bar\phi} = C^{\phi}_{\phi\phi}\,, \qquad d_{\bar\phi\bar\phi} = d_{\phi\phi}\,, \tag{3.72}$$

$$C^{\phi}_{\bar\phi\bar\phi} = C^{\bar\phi}_{\phi\phi} = C^{\bar\phi}_{\phi\bar\phi} = C^{\bar\phi}_{\bar\phi\phi} = C^{\phi}_{\phi\bar\phi} = C^{\phi}_{\bar\phi\phi} = 0\,. \tag{3.73}$$

3.5.3 Two chiral fields

The 24 structure constants involving two chiral fields can be written in terms of just two of these, which we can take to be

$$C^L_{\bar\phi\phi}\,, \quad \text{and} \quad C^L_{\phi\bar\phi}\,. \tag{3.74}$$

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Listing the remaining 22 structure constants:

$$C^{R}_{\phi\bar{\phi}} = C^{L}_{\bar{\phi}\phi}\,, \qquad C^{R}_{\bar{\phi}\phi} = C^{L}_{\phi\bar{\phi}}\,, \tag{3.75}$$

$$C^{\bar{\phi}}_{R\phi} = \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(C^{L}_{\bar{\phi}\phi} + \gamma C^{L}_{\phi\bar{\phi}})\,, \qquad C^{\bar{\phi}}_{L\phi} = \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(\gamma C^{L}_{\bar{\phi}\phi} + C^{L}_{\phi\bar{\phi}})\,, \tag{3.76}$$

$$C^{\phi}_{R\bar{\phi}} = \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(C^{L}_{\phi\bar{\phi}} + \gamma C^{L}_{\bar{\phi}\phi})\,, \qquad C^{\phi}_{L\bar{\phi}} = \frac{d_{\varphi\varphi}}{d_{\phi\phi}}(\gamma C^{L}_{\phi\bar{\phi}} + C^{L}_{\bar{\phi}\phi})\,, \tag{3.77}$$

$$C^{\bar{\phi}}_{\phi R} = \zeta^{-2}\frac{d_{\varphi\varphi}}{d_{\phi\phi}}(C^{L}_{\bar{\phi}\phi} + \gamma C^{L}_{\phi\bar{\phi}})\,, \qquad C^{\bar{\phi}}_{\phi L} = \zeta^{2}\frac{d_{\varphi\varphi}}{d_{\phi\phi}}(\gamma C^{L}_{\bar{\phi}\phi} + C^{L}_{\phi\bar{\phi}})\,, \tag{3.78}$$

$$C^{\phi}_{\bar{\phi}R} = \zeta^{2}\frac{d_{\varphi\varphi}}{d_{\phi\phi}}(C^{L}_{\phi\bar{\phi}} + \gamma C^{L}_{\bar{\phi}\phi})\,, \qquad C^{\phi}_{\bar{\phi}L} = \zeta^{-2}\frac{d_{\varphi\varphi}}{d_{\phi\phi}}(\gamma C^{L}_{\phi\bar{\phi}} + C^{L}_{\bar{\phi}\phi})\,, \tag{3.79}$$

$$C^{R}_{\phi\phi} = C^{L}_{\phi\phi} = C^{R}_{\bar{\phi}\bar{\phi}} = C^{L}_{\bar{\phi}\bar{\phi}} = 0\,, \tag{3.80}$$

$$C^{\phi}_{R\phi} = C^{\phi}_{L\phi} = C^{\bar{\phi}}_{R\bar{\phi}} = C^{\bar{\phi}}_{L\bar{\phi}} = 0\,, \tag{3.81}$$

$$C^{\phi}_{\phi R} = C^{\phi}_{\phi L} = C^{\bar{\phi}}_{\bar{\phi}R} = C^{\bar{\phi}}_{\bar{\phi}L} = 0\,. \tag{3.82}$$

It will be convenient to introduce κ and Γ to parametrise $C^{L}_{\phi\bar{\phi}}$ and $C^{L}_{\bar{\phi}\phi}$ as

$$C^{L}_{\phi\bar{\phi}} = \kappa\Gamma\,, \quad C^{L}_{\bar{\phi}\phi} = \kappa^{-1}\Gamma\,, \quad C^{L}_{\phi\bar{\phi}} = \kappa^{2}C^{L}_{\bar{\phi}\phi}\,. \tag{3.83}$$

It will turn out that Γ is real and non-negative and κ is a pure phase. We note that these two structure constants can be found from the results in [13] — they are related to C_s defined in [13]: eq. (2.19).

3.5.4 One chiral field

The twenty-four structure constants involving just one chiral field can, using the previous identities, be written in terms of just four:

$$C^{R}_{\phi L}\,, \quad C^{R}_{\bar{\phi}L}\,, \quad C^{L}_{\phi R}\,, \quad C^{L}_{\bar{\phi}R}\,. \tag{3.84}$$

We list the remaining twenty constants here for convenience:

$$C^{R}_{L\phi} = \zeta^{-1}C^{R}_{\phi L}\,, \qquad C^{R}_{L\bar{\phi}} = \zeta C^{R}_{\bar{\phi}L}\,, \tag{3.85}$$

$$C^{L}_{R\phi} = \zeta^{-1}C^{L}_{\phi R}\,, \qquad C^{L}_{R\bar{\phi}} = \zeta C^{L}_{\bar{\phi}R}\,, \tag{3.86}$$

$$C^{L}_{\phi L} = -\gamma C^{R}_{\phi L}\,, \qquad C^{R}_{\phi R} = -\gamma C^{L}_{\phi R}\,, \tag{3.87}$$

$$C^{L}_{\bar{\phi}L} = -\gamma C^{R}_{\bar{\phi}L}\,, \qquad C^{R}_{\bar{\phi}R} = -\gamma C^{L}_{\bar{\phi}R}\,, \tag{3.88}$$

$$C^{L}_{L\phi} = \zeta^{-1}C^{L}_{\phi L} = -\gamma\zeta^{-1}\,C^{R}_{\phi L}\,, \qquad C^{L}_{L\bar{\phi}} = \zeta C^{L}_{\bar{\phi}L} = -\gamma\zeta\,C^{R}_{\bar{\phi}L}\,, \tag{3.89}$$

$$C^{R}_{R\phi} = \zeta^{-1}C^{R}_{\phi R} = -\gamma\zeta^{-1}\,C^{L}_{\phi R}\,, \qquad C^{R}_{R\bar{\phi}} = \zeta C^{R}_{\bar{\phi}R} = -\gamma\zeta\,C^{L}_{\bar{\phi}R}\,, \tag{3.90}$$

$$C^{\phi}_{LR} = \frac{1-\gamma^{2}}{\zeta}\frac{d_{\varphi\varphi}}{d_{\phi\phi}}C^{L}_{\phi R}\,, \qquad C^{\bar{\phi}}_{LR} = (1-\gamma^{2})\frac{d_{\varphi\varphi}}{d_{\phi\phi}}C^{R}_{\bar{\phi}L}\,, \tag{3.91}$$

$$C^{\phi}_{RL} = \frac{1-\gamma^{2}}{\zeta^{2}}\frac{d_{\varphi\varphi}}{d_{\phi\phi}}C^{L}_{\phi R}\,, \qquad C^{\bar{\phi}}_{RL} = \zeta\,(1-\gamma^{2})\frac{d_{\varphi\varphi}}{d_{\phi\phi}}C^{R}_{\bar{\phi}L}\,, \tag{3.92}$$

$$C^{\phi}_{LL} = C^{\bar{\phi}}_{LL} = C^{\phi}_{RR} = C^{\bar{\phi}}_{RR} = 0\,. \tag{3.93}$$

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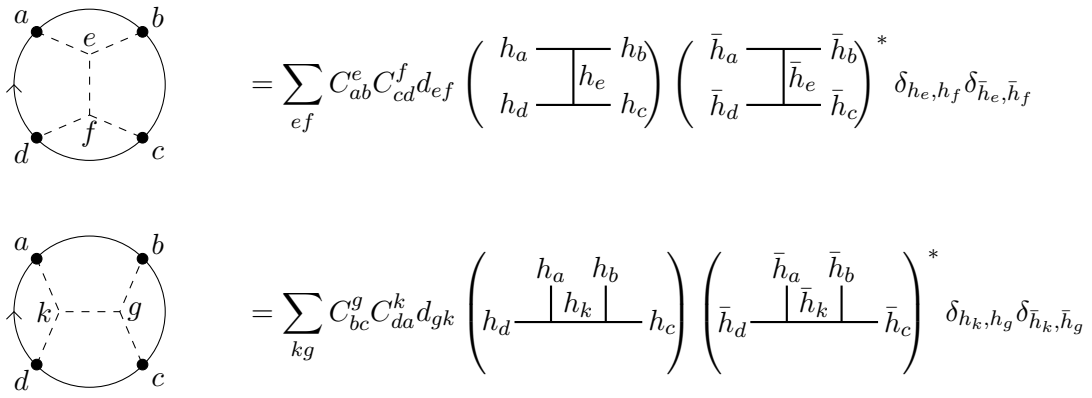


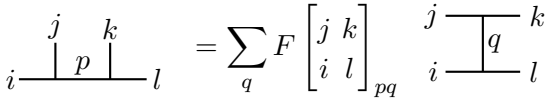
Figure 4. Two ways of calculating a four-point defect field correlation function.

3.6 The four-point function sewing constraints

We will use crossing relations for four point correlation functions to find sewing constraints that will enable us to determined the remaining six structure constants $\{C^L_{\bar{\phi}\phi}, C^L_{\phi\bar{\phi}}, C^R_{\phi L}, C^R_{\bar{\phi}L}, C^L_{\phi R}, C^L_{\bar{\phi}R}\}$.

The four-point function $\langle \Phi_a \Phi_b \Phi_c \Phi_d \rangle$ of fields on a defect can be expressed in terms of conformal blocks in two different ways, as illustrated in figure 4.

The conformal blocks are functions which satisfy the crossing relations [21]



(3.94)

where the F-matrices are known constants, again given explicitly in [21]. Substituting (3.94) into the expressions in figure 4 leads to further sewing constraints that the structure constants must satisfy.

The simplest relations arise when there is only a single channel in both diagrams, i.e. the sum is over a single pair of weights $(h_e, \bar{h}_e)$ and a single pair of weights $(h_g, \bar{h}_g)$. Note that since the space of fields with weights (h, h) is two-dimensional, this does not mean that the OPE has to include only a single field. In all the cases where there is only a single channel, the F-matrix is just the number 1 and so the sewing constraints become just

$$\sum_{e,f} C^e_{ab} C^f_{cd} d_{ef} = \sum_{g,k} C^g_{bc} C^k_{da} d_{gk} \,. \tag{3.95}$$

We now list all the non-zero cases in which the fields a, b, c and d are taken from $\{\phi, \bar{\phi}, \varphi_\alpha\}$ and for which there is only a single intermediate channel in both diagrams, and state the corresponding equations. We will in fact only use the first eight of these, where there is at most one field of weights (h, h) but we list them all for completeness.

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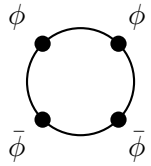
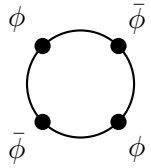
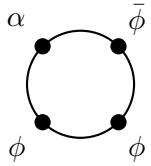
The eight we use are:



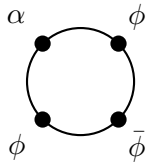
$$d_{\phi\phi}\, d_{\bar{\phi}\bar{\phi}} = \sum_{\alpha,\beta} C^{\alpha}_{\phi\bar{\phi}} C^{\beta}_{\bar{\phi}\phi} d_{\alpha\beta} \tag{3.96}$$



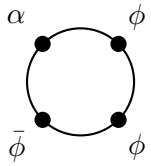
$$\sum_{\alpha,\beta} C^{\alpha}_{\phi\bar{\phi}} C^{\beta}_{\phi\bar{\phi}} d_{\alpha\beta} = \sum_{\alpha,\beta} C^{\alpha}_{\bar{\phi}\phi} C^{\beta}_{\bar{\phi}\phi} d_{\alpha\beta} \tag{3.97}$$



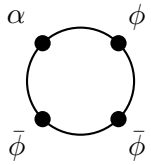
$$C^{\phi}_{\alpha\bar{\phi}} C^{\phi}_{\phi\phi} d_{\phi\phi} = \sum_{\beta,\gamma} C^{\beta}_{\bar{\phi}\phi} C^{\gamma}_{\phi\alpha} d_{\beta\gamma} \tag{3.98}$$



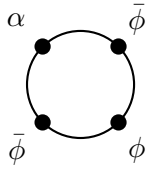
$$\sum_{\beta\gamma} C^{\beta}_{\alpha\phi} C^{\gamma}_{\bar{\phi}\phi} d_{\beta\gamma} = \sum_{\beta,\gamma} C^{\beta}_{\phi\bar{\phi}} C^{\gamma}_{\phi\alpha} d_{\beta\gamma} \tag{3.99}$$



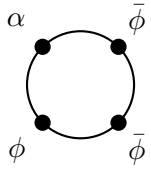
$$\sum_{\beta\gamma} C^{\beta}_{\alpha\phi} C^{\gamma}_{\phi\bar{\phi}} d_{\beta\gamma} = C^{\phi}_{\phi\phi} C^{\phi}_{\bar{\phi}\alpha} d_{\phi\phi} \tag{3.100}$$



$$C^{\bar{\phi}}_{\alpha\phi} C^{\bar{\phi}}_{\bar{\phi}\bar{\phi}} d_{\bar{\phi}\bar{\phi}} = \sum_{\beta,\gamma} C^{\beta}_{\phi\bar{\phi}} C^{\gamma}_{\bar{\phi}\alpha} d_{\beta\gamma} \tag{3.101}$$



$$\sum_{\beta\gamma} C^{\beta}_{\alpha\bar{\phi}} C^{\gamma}_{\phi\bar{\phi}} d_{\beta\gamma} = \sum_{\beta,\gamma} C^{\beta}_{\bar{\phi}\phi} C^{\gamma}_{\bar{\phi}\alpha} d_{\beta\gamma} \tag{3.102}$$



$$\sum_{\beta\gamma} C^{\beta}_{\alpha\bar{\phi}} C^{\gamma}_{\bar{\phi}\phi} d_{\beta\gamma} = C^{\bar{\phi}}_{\bar{\phi}\bar{\phi}} C^{\bar{\phi}}_{\phi\alpha} d_{\bar{\phi}\bar{\phi}} \tag{3.103}$$

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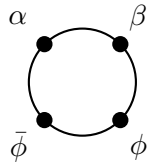
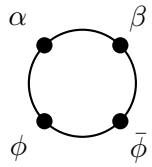
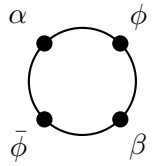
The remaining three which include two fields of type φ_α but still only have a single intermediate channel are:



$$\sum_{\gamma\epsilon} C^{\gamma}_{\alpha\beta} C^{\epsilon}_{\phi\bar{\phi}} d_{\gamma\epsilon} = \sum_{\gamma\epsilon} C^{\gamma}_{\beta\phi} C^{\epsilon}_{\bar{\phi}\alpha} d_{\gamma\epsilon} \tag{3.104}$$



$$\sum_{\gamma\epsilon} C^{\gamma}_{\alpha\beta} C^{\epsilon}_{\bar{\phi}\phi} d_{\gamma\epsilon} = \sum_{\gamma\epsilon} C^{\gamma}_{\beta\bar{\phi}} C^{\epsilon}_{\phi\alpha} d_{\gamma\epsilon} \tag{3.105}$$



$$\sum_{\gamma\epsilon} C^{\gamma}_{\alpha\phi} C^{\epsilon}_{\beta\bar{\phi}} d_{\gamma\epsilon} = \sum_{\gamma\epsilon} C^{\gamma}_{\phi\beta} C^{\epsilon}_{\bar{\phi}\alpha} d_{\gamma\epsilon} \tag{3.106}$$

3.7 Analysis of the sewing constraints

We need to use only the first eight relations. We consider these in turn:

- Equation (3.96)

  Written out in full, this is

  $$d_{\phi\phi} d_{\bar{\phi}\bar{\phi}} = C^{L}_{\phi\bar{\phi}} C^{L}_{\bar{\phi}\phi} d_{LL} + C^{L}_{\phi\bar{\phi}} C^{R}_{\bar{\phi}\phi} d_{LR} + C^{R}_{\phi\bar{\phi}} C^{L}_{\bar{\phi}\phi} d_{RL} + C^{R}_{\phi\bar{\phi}} C^{R}_{\bar{\phi}\phi} d_{RR} \,. \tag{3.107}$$

  Using $C^{L}_{\phi\bar{\phi}} = C^{R}_{\bar{\phi}\phi} = \kappa\Gamma$ and $C^{R}_{\phi\bar{\phi}} = C^{L}_{\bar{\phi}\phi} = \kappa^{-1}\Gamma$, together with $d_{LR} = d_{RL} = \gamma d_{\varphi\varphi}$, and $d_{\phi\phi} = d_{\bar{\phi}\bar{\phi}}$, this becomes

  $$\frac{d^2_{\phi\phi}}{d_{\varphi\varphi}} = \Gamma^2 (2 + \gamma\kappa^2 + \gamma\kappa^{-2}) \,, \tag{3.108}$$

  or

  $$\Gamma = \sqrt{\frac{d^2_{\phi\phi}}{d_{\varphi\varphi}\,(2 + \gamma\kappa^2 + \gamma\kappa^{-2})}} \,. \tag{3.109}$$

- Equation (3.97)

  This is

  $$\begin{aligned} &C^{L}_{\phi\bar{\phi}} C^{L}_{\phi\bar{\phi}} d_{LL} + C^{L}_{\phi\bar{\phi}} C^{R}_{\phi\bar{\phi}} d_{LR} + C^{R}_{\phi\bar{\phi}} C^{L}_{\phi\bar{\phi}} d_{RL} + C^{R}_{\phi\bar{\phi}} C^{R}_{\phi\bar{\phi}} d_{RR} \\ &= C^{L}_{\bar{\phi}\phi} C^{L}_{\bar{\phi}\phi} d_{LL} + C^{L}_{\bar{\phi}\phi} C^{R}_{\bar{\phi}\phi} d_{LR} + C^{R}_{\bar{\phi}\phi} C^{L}_{\bar{\phi}\phi} d_{RL} + C^{R}_{\bar{\phi}\phi} C^{R}_{\bar{\phi}\phi} d_{RR} \,, \end{aligned} \tag{3.110}$$

  which is satisfied identically.

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- Equation (3.98)

  This leads to two equations: for $\alpha = L$:

$$C^{\phi}_{L\bar{\phi}} C^{\phi}_{\phi\phi} d_{\phi\phi} = C^{L}_{\bar{\phi}\phi} C^{L}_{\phi L} d_{LL} + C^{L}_{\bar{\phi}\phi} C^{R}_{\phi L} d_{LR} + C^{R}_{\bar{\phi}\phi} C^{L}_{\phi L} d_{RL} + C^{R}_{\bar{\phi}\phi} C^{R}_{\phi L} d_{RR} \,, \tag{3.111}$$

  and for $\alpha = R$:

$$C^{\phi}_{R\bar{\phi}} C^{\phi}_{\phi\phi} d_{\phi\phi} = C^{L}_{\bar{\phi}\phi} C^{L}_{\phi R} d_{LL} + C^{L}_{\bar{\phi}\phi} C^{R}_{\phi R} d_{LR} + C^{R}_{\bar{\phi}\phi} C^{L}_{\phi R} d_{RL} + C^{R}_{\bar{\phi}\phi} C^{R}_{\phi R} d_{RR} \,. \tag{3.112}$$

  The first equation becomes:

$$(\gamma C^{L}_{\phi\bar{\phi}} + C^{L}_{\bar{\phi}\phi}) C^{\phi}_{\phi\phi} = C^{R}_{\phi L} C^{L}_{\phi\bar{\phi}} \left(1 - \gamma^2\right) \,, \tag{3.113}$$

  or

$$C^{R}_{\phi L} = \frac{1 + \kappa^2 \gamma}{\kappa^2 (1 - \gamma^2)} C^{\phi}_{\phi\phi} \,. \tag{3.114}$$

  The second equation implies

$$C^{L}_{\phi R} = \frac{\kappa^2 + \gamma}{(1 - \gamma^2)} C^{\phi}_{\phi\phi} \,. \tag{3.115}$$

- Equation (3.99)

  These two equations imply

$$\kappa^2 = \zeta = \exp(i\pi h) \,. \tag{3.116}$$

  (We will not need to fix the sign of κ as only κ^2 appears in our final answers.)

- Equation (3.100)

  These equations imply (for $\alpha = L$)

$$C^{R}_{\phi L} = \frac{1 + \kappa^2 \gamma}{\zeta (1 - \gamma^2)} C^{\phi}_{\phi\phi} \,, \tag{3.117}$$

  and (for $\alpha = R$)

$$C^{L}_{\phi R} = \frac{\zeta^2}{\kappa} \frac{\gamma + \kappa^2}{(1 - \gamma^2)} C^{\phi}_{\phi\phi} \,, \tag{3.118}$$

  which are consistent with the results so far.

- Equation (3.101)

  These two equations lead to ($\alpha = L$):

$$C^{R}_{\bar{\phi}L} = \frac{\gamma + \kappa^2}{1 - \gamma^2} C^{\phi}_{\phi\phi} \,, \tag{3.119}$$

  and (with $\alpha = R$)

$$C^{L}_{\bar{\phi}R} = \frac{1 + \gamma \kappa^2}{\kappa^2 (1 - \gamma^2)} C^{\phi}_{\phi\phi} \,. \tag{3.120}$$

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Together, these imply

$$C^R_{\bar\phi L} = C^L_{\phi R} \quad \text{and} \quad C^L_{\bar\phi R} = C^R_{\phi L}\,. \tag{3.121}$$

This completes the derivation of the defect structure constants which in each case have been found in terms of the bulk and boundary CFT data without reference to the particular values in the minimal models, relying only on the form of the OPEs, the fusion rules and the restriction of the crossing relations to four-point functions for which there is a single intermediate channel for which $F = 1$.

These constants have already been worked out in the case of the Lee-Yang model in [14] and we agree with those found previously (apart from a typo in [14], where it should $\rho = \exp(i\pi/10)$).

The remaining crossing relations (3.104)–(3.106) are not needed for the derivation of the structure constants but we have checked that they hold.

4 The integrals

We want to calculate the leading term in the expansion (2.10), that is

$$I = \frac{1}{4}\lambda^2\bar\lambda^2 \int \mathrm{d}x\,\mathrm{d}x'\,\mathrm{d}y\,\mathrm{d}y' \langle\, T(iY)\,\overline{T}(iY)\,\phi(x)\phi(x')\bar\phi(y)\bar\phi(y')\,\rangle\,. \tag{4.1}$$

The correlation function has the same functional form whatever the order of the fields, but a different constant depending on the order of the insertions. We can restrict to $x < x'$ and $y < y'$ to get

$$I = (\lambda\bar\lambda)^2 \left\langle\, T(i)\bar T(i) \int_{x<x'\,,\,y<y'} \mathrm{d}x\,\mathrm{d}x'\,\mathrm{d}y\,\mathrm{d}y'\,\phi(x)\phi(x')\bar\phi(y)\bar\phi(y') \,\right\rangle_{D_{r2}}\,. \tag{4.2}$$

This correlation function is

$$\langle\, T(i)\,\bar T(i)\,\phi(x)\phi(x')\bar\phi(y)\bar\phi(y')\,\rangle = \Delta\, h^2\, \frac{(x'-x)^{2-2h}\,(y'-y)^{2-2h}}{(i-x)^2(i-x')^2(i+y)^2(i+y')^2}\,, \tag{4.3}$$

where the constant Δ depends on the order of the field insertions as in table 2.

The values Δ_i are

$$\Delta_1 = d_{\phi\phi}\,d_{\bar\phi\bar\phi} = (d_{\phi\phi})^2\,, \tag{4.4}$$

$$\Delta_2 = d_{\alpha\beta}\,C^\alpha_{\phi\bar\phi}C^\beta_{\phi\bar\phi} = (d_{\phi\phi})^2\,\frac{2\gamma+\kappa^2+\kappa^{-2}}{2+\gamma\kappa^2+\gamma\kappa^{-2}} = (d_{\phi\phi})^2\,\frac{\gamma+\cos(\pi h)}{1+\gamma\cos(\pi h)}\,. \tag{4.5}$$

We only need to evaluate three of these integrations, the other three being given by complex conjugation. Furthermore, we only need the leading order term in y in the correlation function,

$$\langle\, T(i)\bar T(i)\phi(x)\phi(x')\bar\phi(y)\bar\phi(y')\,\rangle_{D_{r2}} = \frac{\Delta}{(i-x)^2(i-x')^2(i+y)^2(i+y')^2} + O(y)\,. \tag{4.6}$$

Integration region	Order of fields	Value of Δ
$x < x' < y < y'$	$\phi\phi\bar{\phi}\bar{\phi}$	Δ_1
$x < y < x' < y'$	$\phi\bar{\phi}\phi\bar{\phi}$	Δ_2
$x < y < y' < x'$	$\phi\bar{\phi}\bar{\phi}\phi$	Δ_1
$y < x < x' < y'$	$\bar{\phi}\phi\phi\bar{\phi}$	Δ_1
$y < x < y' < x'$	$\bar{\phi}\phi\bar{\phi}\phi$	Δ_2
$y < y' < x < x'$	$\bar{\phi}\bar{\phi}\phi\phi$	Δ_1

Table 2. The coefficient in the four-point function (4.3).

Integration region	Order of fields	Value of the integral
$x < x' < y < y'$	$\phi\phi\bar{\phi}\bar{\phi}$	$-\frac{3\pi i}{16}\Delta_1$
$x < y < x' < y'$	$\phi\bar{\phi}\phi\bar{\phi}$	$-\frac{\pi^2+3\pi i}{8}\Delta_2$
$x < y < y' < x'$	$\phi\bar{\phi}\bar{\phi}\phi$	$\frac{\pi^2}{8}\Delta_1$
$y < x < x' < y'$	$\bar{\phi}\phi\phi\bar{\phi}$	$\frac{\pi^2}{8}\Delta_1$
$y < x < y' < x'$	$\bar{\phi}\phi\bar{\phi}\phi$	$-\frac{\pi^2-3\pi i}{8}\Delta_2$
$y < y' < x < x'$	$\bar{\phi}\bar{\phi}\phi\phi$	$\frac{3\pi i}{16}\Delta_1$

Table 3. The integrals.

The results are given in table 3. Adding all six together, we get

$$\begin{aligned} I &= (\lambda\bar{\lambda})^2 \int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{d}x'\,\mathrm{d}y\,\mathrm{d}y'\ \langle\, T(i)\bar{T}(i)\phi(x)\phi(x')\bar{\phi}(y)\bar{\phi}(y')\,\rangle_{D_{r2}} \\ &= (\lambda\bar{\lambda})^2 \left[\frac{\pi^2}{4}(\Delta_1-\Delta_2)+O(y)\right] \\ &= (\lambda\bar{\lambda})^2 \left[\frac{\pi^2(d_{\phi\phi})^2}{2}\,\frac{(1-\gamma)\sin^2(\pi h/2)}{1+\gamma\cos(\pi h)}+O(y)\right]. \end{aligned} \tag{4.7}$$

5 The value of the reflection coefficient for the defect C

We now put the various terms together to find the value of $\mathcal{R}$ at the fixed point (λ^*, λ^*),

$$\mathcal{R} = \frac{\langle T^1\overline{T}^1 + T^2\overline{T}^2\rangle}{\langle (T^1+\overline{T}^2)(\overline{T}^1+T^2)\rangle}\,. \tag{5.1}$$

The leading term in the numerator is $2I$ and leading term in the denominator is $c/16$, so that at the fixed point $(\lambda^*, \bar{\lambda}^*)$,

$$\mathcal{R}^* = \frac{16\pi^2(d_{\phi\phi})^2}{c}\,\frac{(1-\gamma)\sin(h\pi/2)^2}{1+\gamma\cos(h\pi)}(\lambda^*\bar{\lambda}^*)^2+O(y^5)\,. \tag{5.2}$$

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So far, this result only depends on the form of the operator product algebra of the boundary and bulk fields and on the operator content of fields on the defect, and to that extent is not specific to the minimal models. We can now use the values of h (3.1), the combination $(C^{\phi}_{\phi\phi})^2/d_{\phi\phi}$ (3.5), and the value of γ (3.27) to obtain the result for the minimal model defects of type $(r, 2)$ perturbed by the field ϕ_{13}.

We will now give directly the expansion in $y = 1 - h$ of the various constants in (5.2) together with the result for $\mathcal{R}^*$: with $t = 1 - y/2$, we have

$$h = 2t - 1 = 1 - y\,, \qquad c = 13 - 6t - 6/t = 1 - \frac{3y^2}{2} + O(y^3)\,,$$
$$\gamma = 2\cos(2\pi t) - 1 = 1 + O(y^2)\,, \quad \frac{(C^{\phi}_{\phi\phi})^2}{d_{\phi\phi}} = \frac{8}{3} + O(y)\,, \quad \lambda^* = \bar{\lambda}^* = \frac{y}{C^{\phi}_{\phi\phi}}\,, \tag{5.3}$$

and so, at the fixed point,

$$\mathcal{R}^* = \frac{9\pi^2 y^4}{8} + O(y^5)\,. \tag{5.4}$$

6 Comparison with known results

There are very few known exact results for non-topological non-factorising defects in minimal models with which we can compare our formula (5.4).

The simplest model is the Ising model, the (A_3, A_4) model with $c = 1/2$, $h = 1/2$, $y = 1.2$, and which is very far from the perturbative regime. For this model, the end point of the perturbation (2.2) is known exactly [2]. There is a 1 parameter family of flows with $\lambda = \mu\cos(\alpha)$ and $\bar{\lambda} = \mu\sin(\alpha)$, and for each value of α the end point of the flow is a known conformal defect with

$$\mathcal{R} = \sin^2(2\alpha)\,. \tag{6.1}$$

If the perturbation is calculated in a scheme in which the fixed point is at $\mu = \mu^*$ then this result can also be written

$$\mathcal{R} = 4\frac{(\lambda^*\bar{\lambda}^*)^2}{(\mu^*)^4}\,. \tag{6.2}$$

Although this looks very much like (5.4), they cannot be compared directly since the three-point coupling $C^{\phi}_{\phi\phi}$ is identically zero in the Ising model and our calculation is not valid in this model.

The only other minimal model for which exact results are known is the tri-critical Ising model. Non-topological non-factorising defects were constructed for this model in [7]. The defects found in [7] are still rather special, and are likely to be sums of more fundamental non-topological non-factorising defects of the same value of $\mathcal{R}$.

This model has $h = 3/5, y = 2/5$ and is again far from the perturbative regime but we can at least compare the perturbative calculation with the values of $\mathcal{R}$ found in [7]. We can also calculate the value of the defect entropy, g, of the perturbative defect and of the exact defects as a guide to how reliable the perturbative calculation is this far from the small y regime. Let us consider just the perturbations of the defect $D_{1,2}$, as in [6]. The calculation of g of the perturbed defect is given in [6] (with the choice $d_{\phi\phi} = 1$) as

$$\log(g(\lambda, \bar{\lambda})) = \log(g_{1,2}) - \pi^2 y(\lambda^2 + \bar{\lambda}^2) + \frac{4\sqrt{2}\pi^2}{3\sqrt{3}}(\lambda^3 + \bar{\lambda}^3) + O(y^4)\,. \tag{6.3}$$

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As shown in [6], the perturbative calculation of the change in the defect entropy agrees exactly with the perturbative expansions of the entropy at the UV and IR fixed points for the case of the flow between topological defects $D_{1,2} \to D_{2,1}$ which is the perturbative fixed point at $(y\sqrt{3/8}, 0)$. The exact value for the defects $D_{1,2}$ and $D_{2,1}$ are

$$\begin{aligned} g_{1,2} &= -2\cos(\pi t)\,, \quad \log(g_{1,2}) = \log(2) - \frac{\pi^2 y^2}{8} + O(y^4)\,, \\ g_{2,1} &= -2\cos(\pi/t)\,, \quad \log(g_{1,2}) = \log(2) - \frac{\pi^2 y^2}{8} - \frac{\pi^2 y^3}{8} + O(y^4)\,, \end{aligned} \tag{6.4}$$

and so

$$\log(g_{2,1}) - \log(g_{1,2}) = \log\cos(\pi/t) - \log\cos(\pi t) = -\frac{\pi^2 y^3}{8} + O(y^4)\,, \tag{6.5}$$

in perfect agreement with (6.3) at the fixed point $(y\sqrt{3/8}, 0)$. When we look at the numerical values for the tri-critical Ising model with $y = 2/5$, however, the agreement is not so good. The exact values are

$$g_{1,2} = \frac{1+\sqrt{5}}{2}\,, \quad \log(g_{1,2}) = 0.48121\ldots\,, \quad g_{2,1} = \sqrt{2}\,, \quad \log(g_{2,1}) = 0.34657\ldots\,, \tag{6.6}$$

$$\log(g_{2,1}) - \log(g_{1,2}) = -0.13464\ldots\,, \tag{6.7}$$

whereas the perturbation theory calculation of $\log(g_{2,1})$ to third order, (6.5), gives

$$\log(g_{2,1})_{\text{3rd order}} - \log(g_{1,2}) = -\frac{\pi^2 y^3}{8} = -0.78956\ldots\,. \tag{6.8}$$

The g value correctly decreases under the flow but the leading order term in the change, $-\pi^2 y^3/8$, only contributes 60% of the full change in $\log g$.

Let's now consider the perturbative predictions for the defect C in the tri-critical Ising model which is believed to be at the fixed point (λ^*, λ^*). The predictions of (6.3) and (5.4) for g_C and $\mathcal{R} - C$ are

$$\log(g_C^{\text{pert}}) = \log(g_{1,2}) - 2\frac{\pi^2 y^3}{8} = 0.3233\ldots \tag{6.9}$$

$$g_C^{\text{pert}} = 1.3817\ldots \tag{6.10}$$

$$\mathcal{R}_C^{\text{pert}} = \frac{9\pi^2 y^4}{8} = 0.2842\ldots \tag{6.11}$$

This can be compared with the two distinct values of $\mathcal{R}$ for non-topological non-factorised defects found in [7], which are

$$\mathcal{R}_1 = \frac{\sqrt{3}-1}{2} = 0.366\ldots \quad \text{and} \quad \mathcal{R}_2 = \frac{3-\sqrt{3}}{2} = 0.633\ldots\,. \tag{6.12}$$

Given the known size of the error in the perturbative calculation of g, it is completely plausible that the value of $\mathcal{R}$ for the defect C is indeed $\mathcal{R}_1 = (\sqrt{3}-1)/2$, but it is harder to say anything stronger than that.

We know that none of the defects found in [7] can be equal to C since they all have g greater than 2. If there is a direct relation, then the most likely explanation is that the defects in [7] are composite defects formed of two [or more] defects with the same values of $\mathcal{R}$.

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The tri-critical Ising model defects constructed in [7] are given through the action of interface operators on defects in a supersymmetric theory. The action of the interfaces is to double the g value of the defect. The g values of the supersymmetric defects are given in table 4.3 of [7], and there are only two non-topological non-factorising supersymmetric defects with $g < \sqrt{2}$, which are constructed from boundary states $||(1,3)_{NS}\rangle\rangle$ and $||(1,3)_{\widetilde{NS}}\rangle\rangle$ with g value

$$g_{\rm SUSY} = (15)^{-1/4} \left(\frac{(3+\sqrt{3})(\sqrt{5}-1)}{2(\sqrt{3}-1)} \right)^{1/2} = 1.0156\ldots \tag{6.13}$$

The action of the interface operators will then give defects in the tri-critical Ising model with $g_{\rm TCIM} = 2g_{\rm SUSY}$. the conjecture would be that these defects are themselves composite defects of fundamental non-topological non-factorising defects with g value half that of $g_{\rm TCIM}$, that is $g_{\rm SUSY}$. The upshot is that we could conjecture the existence of non-topological non-factorising defects in the tri-critical Ising model, related to the supersymmetric defects found in [7], and with

$$g_{\rm TCIM}^{\rm conjectured} = 1.0156\ldots \tag{6.14}$$

It is certainly possible that this is the same as g_C but the perturbative calculation of g_C (6.10) is again not accurate enough to say anything definite about this identification.

To summarise, the perturbative calculations of the g value of $\mathcal{R}$ value of the defect C are

$$g_C^{\rm pert} = 1.3817\ldots\,, \quad \mathcal{R}_C^{\rm pert} = 0.2842\ldots\,. \tag{6.15}$$

The most likely candidate so far for an exact description of this defect is a conjectured component of a defect found in [7] with

$$g_{\rm TCIM}^{\rm conjectured} = 1.0156\ldots\,, \quad \mathcal{R}_1 = 0.366\ldots\,, \tag{6.16}$$

but the calculations are not accurate enough to say anything more than that is a plausible conjecture and certainly not ruled out.

7 Conclusions

We have calculated the leading term in the perturbative expansion of the reflection coefficient for the defect of type $(r,2)$ in a minimal model. It is believed that a non-trivial conformal defect can be found as a perturbative fixed point of the renormalisation group equations, where the expansion parameter is the usual $y = 2(1-t)$ which for a unitary minimal model $M_{p,p+1}$ is $y = 2/(1+p)$.

We have recently found new non-trivial conformal defects in the tri-critical Ising model [7] and it is possible that these are related to the conformal defects found by perturbation theory but the value of $y = 2/5$ is large and so the perturbative calculations are not expected to be very accurate. We have checked, and the values of g and $\mathcal{R}$ are close enough not to rule this out. It would of course be good to extend the calculation of $\mathcal{R}$ to next-to-leading order where there are UV divergences to be regulated, but so far we have not yet managed this.

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We have also calculated defect structure constants for various fields on defects of type $(r, 2)$ extending the results of [14]. These results are not complete — they do not include all fields — but it would be good to check that these constants in fact agree with the general results of [12] where the same constants were constructed using topological field theory methods.

The methods and results here are not specific to the Virasoro minimal model $(r, 2)$ defect as they only rely on the properties of the field content and fusion rules. While we think the $(r, 2)$ defects are the only Virasoro defects to which these calculations are applicable, they could in principle have wider applicability.

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A The Virasoro minimal models

The Virasoro minimal models occur for $c \equiv c(p, q)$ where p, q are coprime positive integers greater than 1. It is useful to define $t = p/q$. c is given by

$$c(p, q) = 13 - 6t - 6/t\,. \tag{A.1}$$

There are $(p-1)(q-1)/2$ minimal representations labelled by integers (r, s) with $1 \leq r < p$, $1 \leq s < q$ with conformal weights

$$h_{r,s} = \frac{(rq - sp)^2 - (p - q)^2}{4pq} = \frac{r^2 - 1}{4t} + \frac{s^2 - 1}{4}t - \frac{rs - 1}{2}\,. \tag{A.2}$$

The modular S-matrix is

$$S_{(r,s),(r',s')} = (-1)^{1+rs'+r's}\sqrt{\frac{8}{pq}}\sin(\pi rr'/t)\sin(\pi ss't)\,. \tag{A.3}$$

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